

Exact ABJM Partition Function from TBA

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Abstract

We report on the exact computation of the S^3 partition function of $U(N)_k \times U(N)_{-k}$ ABJM theory for $k = 1, N = 1, \dots, 19$. The result is a polynomial in π^{-1} with rational coefficients. As an application of our results we numerically determine the coefficient of the membrane 1-instanton correction to the partition function.

1 Introduction and Summary

It has recently been discovered that the partition function of a Chern–Simons–matter (CSM) theory with $\mathcal{N} \geq 2$ supersymmetry on a three-dimensional sphere reduces to a matrix integral [1, 2, 3]. These matrix integrals are powerful quantitative tools to analyze CSM theories, and has lead to a number of important results, including the successful derivation of the $N^{3/2}$ behavior [4] and various precise checks of the $\text{AdS}_4/\text{CFT}_3$ correspondence (see [5, 6, 7] and subsequent works).

In this paper we study the CSM theory with the highest amount of supersymmetry ($\mathcal{N} \geq 6$), namely the ABJM theory [8]. It has gauge group $U(N)_k \times U(N)_{-k}$, where k is the level of the Chern–Simons term. Since ABJM theory is the worldvolume theory of multiple M2-branes, it is natural to ask if we could extract any useful data about M-theory from the three-sphere partition function of the ABJM theory.

In M-theory we have non-perturbative corrections from membrane instantons. This is reflected in the three-sphere partition function as an expansion of the terms of order $e^{-\sqrt{N/k}}$ [9]. However, this expansion is not directly captured in most of the previous analysis of the three-sphere partition function, where we take the t’ Hooft limit N, k large with N/k kept finite. Instead we need to take the M-theory limit, with N large and k kept finite. The leading N contribution in this limit is determined by [7] and the all order $1/N$ expansion in [10]. Moreover the paper [10] discuss the non-perturbative instanton correction in an expansion around $k = 0$. However the for the most interesting case of k finite, the general results on the non-perturbative corrections are still lacking. To answer this question it will be of great help to systematically compute the behavior of the three-sphere partition function for finite N and k .

In this brief note we report on the exact computation of S^3 partition function $Z(N)$ of the $k = 1$ ($\mathcal{N} = 8$) ABJM theory for $N = 1, \dots, 19$, based on the Fermi gas approach of [10] and the TBA-like equations of [11, 12].¹

¹See [13] for exact computation for $N = 2$ and general k , and [14] for numerical calculations.

Our results are given as follows:

$$\begin{aligned}
Z(1) &= \frac{1}{4}, & Z(2) &= \frac{1}{16\pi}, & Z(3) &= \frac{-3 + \pi}{64\pi}, & Z(4) &= \frac{-\pi^2 + 10}{1024\pi^2}, & Z(5) &= \frac{26 + 20\pi - 9\pi^2}{4096\pi^2}, \\
Z(6) &= \frac{78 - 121\pi^2 + 36\pi^3}{147456\pi^3}, & Z(7) &= \frac{-126 + 174\pi + 193\pi^2 - 75\pi^3}{196608\pi^3}, \\
Z(8) &= \frac{876 - 4148\pi^2 - 2016\pi^3 + 1053\pi^4}{18874368\pi^4}, & Z(9) &= \frac{4140 + 8880\pi - 15348\pi^2 - 13480\pi^3 + 5517\pi^4}{75497472\pi^4}, \\
Z(10) &= \frac{16860 - 136700\pi^2 + 190800\pi^3 + 207413\pi^4 - 81000\pi^5}{7549747200\pi^5}, \\
Z(11) &= \frac{-122580 + 381900\pi + 837300\pi^2 - 1289300\pi^3 - 1091439\pi^4 + 447525\pi^5}{30198988800\pi^5}, \\
Z(12) &= \frac{626760 - 8856300\pi^2 - 18446400\pi^3 + 35287138\pi^4 + 30204000\pi^5 - 12504375\pi^6}{4348654387200\pi^6}, \\
Z(13) &= \frac{1563480 + 6714000\pi - 17252100\pi^2 - 40746000\pi^3 + 49141894\pi^4 + 45780780\pi^5 - 18083925\pi^6}{5798205849600\pi^6}, \\
Z(14) &= (21382200 - 421152060\pi^2 + 1918350000\pi^3 + 2614227910\pi^4 - \\
&\quad - 5654854800\pi^5 - 3965159223\pi^6 + 1732468500\pi^7) / (3409345039564800\pi^7), \\
Z(15) &= (-222059880 + 1271579400\pi + 3613033620\pi^2 - 12266517900\pi^3 - 17757814914\pi^4 + \\
&\quad + 28941378130\pi^5 + 21727092861\pi^6 - 9162734175\pi^7) / (13637380158259200\pi^7), \\
Z(16) &= (288454320 - 8196414240\pi^2 - 54540622080\pi^3 + 83379537976\pi^4 + 337956998400\pi^5 - 310977507352\pi^6 - \\
&\quad - 354450849984\pi^7 + 132764935275\pi^8) / (872792330128588800\pi^8), \\
Z(17) &= (3171011760 + 23555952000\pi - 71723746080\pi^2 - 333199608000\pi^3 + 542885550648\pi^4 + 1355261623520\pi^5 - \\
&\quad - 1384280129304\pi^6 - 1337978574000\pi^7 + 518021476875\pi^8) / (3491169320514355200\pi^8), \\
Z(18) &= (4970745360 - 180631896480\pi^2 + 2270514395520\pi^3 + 2444801550408\pi^4 - \\
&\quad - 18251132155200\pi^5 - 13590443330584\pi^6 + 35949047139936\pi^7 + \\
&\quad + 20671882502409\pi^8 - 9607077219600\pi^9) / (377046286615550361600\pi^9), \\
Z(19) &= (-2636096400 + 24895105200\pi + 79219113120\pi^2 - 487774106400\pi^3 - \\
&\quad - 852843285000\pi^4 + 3053792290360\pi^5 + 3630439618136\pi^6 - 6122444513560\pi^7 - \\
&\quad - 4288974330849\pi^8 + 1840384320075\pi^9) / (55858709128229683200\pi^9).
\end{aligned} \tag{1}$$

Section 2 of this paper is devoted to the derivation of this result. Similar methods could be applied to $k > 1$. It would be interesting to find an analytic expression for general k and N .

The knowledge of the exact values of $Z(N)$ in this paper allows one to perform various numerical tests with high precision. As an example, we compute in Section 3 the coefficient of the membrane 1-instanton contribution to the partition function (see (30)).

Note: During the preparation of this manuscript we received a paper [16], which has substantial overlap with our paper. The paper contains the exact results up to $N = 9$, which is consistent with ours.

2 Derivation

Let us consider the grand canonical partition function

$$\Xi(z) = 1 + \sum_{N \geq 1} Z(N) z^N . \quad (2)$$

As is shown in [10], this is given by a Fredholm determinant

$$\Xi(z) = \text{Det} \left(1 + \frac{z \hat{K}}{4\pi} \right) , \quad (3)$$

with \hat{K} defined by an integral kernel

$$K(x, y) := \langle x | \hat{K} | y \rangle = \frac{e^{-u(x) - u(y)}}{\cosh \left(\frac{x-y}{2} \right)} , \quad (4)$$

and

$$u(x) = \frac{1}{2} \log \left(2 \cosh \frac{kx}{2} \right) . \quad (5)$$

In practice, it is useful to use the following relation:

$$\Xi(z) = \exp \left(\text{Tr} \log \left(1 + \frac{z \hat{K}}{4\pi} \right) \right) = \exp \left(- \sum_{\ell} Z_{\ell} \frac{(-z)^{\ell}}{\ell} \right) , \quad (6)$$

with

$$Z_{\ell} = \frac{1}{(4\pi)^{\ell}} \text{Tr} (\hat{K}^{\ell}) = \frac{1}{(4\pi)^{\ell}} \int dx_1 \dots dx_{\ell} K(x_1, x_2) K(x_2, x_3) \dots K(x_{\ell-1}, x_{\ell}) K(x_{\ell}, x_1) . \quad (7)$$

The problem thus reduces to the computation of Z_{ℓ} .

Let us define the kernel for the operator $\hat{K}(I - \lambda^2 \hat{K}^2)^{-1}$ by $R_+(x, y)$ and for $\lambda \hat{K}^2(I - \lambda^2 \hat{K}^2)^{-1}$ by $R_-(x, y)$, respectively. We also denote $R_+(x) := R_+(x, x)$, $R_-(x) := R_-(x, x)$. As is clear from the definition, the integral of $R_{\pm}(x)$ gives Z_{ℓ} :

$$\frac{1}{4\pi} \int dx R_+(x) = \sum_{n \geq 0} (4\pi \lambda)^{2n} Z_{2n+1} , \quad \frac{1}{4\pi} \int dx R_-(x) = \sum_{n \geq 0} (4\pi \lambda)^{2n+1} Z_{2n+2} . \quad (8)$$

Let us further define $\epsilon(\theta), \eta(\theta)$ by

$$e^{-\epsilon(\theta)} = R_+(\theta) , \quad \eta(\theta) = 2\lambda \int_{-\infty}^{\infty} \frac{e^{-\epsilon(\theta')}}{\cosh(\theta - \theta')} d\theta' . \quad (9)$$

It was conjectured in [11] and later proven in [12] that these functions satisfy the following two TBA-like equations:

$$\epsilon(\theta) = 2u(\theta) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1 + \eta^2(\theta'))}{\cosh(\theta - \theta')} d\theta' , \quad (10)$$

$$R_-(\theta) = \frac{1}{\pi} R_+(\theta) \int_{-\infty}^{\infty} \frac{\arctan \eta(\theta')}{\cosh^2(\theta - \theta')} d\theta' . \quad (11)$$

Let us define

$$\epsilon(\theta) = \sum_{n \geq 0} \epsilon_n(\theta) \lambda^{2n} , \quad \eta(\theta) = \sum_{n \geq 0} \eta_n(\theta) \lambda^{2n+1} , \quad (12)$$

$$R_+(\theta) = \sum_{n \geq 0} R_{+,n}(\theta) \lambda^{2n} , \quad R_-(\theta) = \sum_{n \geq 0} R_{-,n}(\theta) \lambda^{2n+1} . \quad (13)$$

Suppose $\epsilon_n(\theta)$, $n = 0 \dots j$ are known. We can then find $\eta_n(\theta)$, $n = 0 \dots j$ by performing the integration in (9), and then $\epsilon_{j+1}(\theta)$ from (10). Thus one can solve the TBA-like equations recursively, order by order in λ , starting from

$$\epsilon_0(\theta) = 2u(\theta) . \quad (14)$$

Once we know $\epsilon_n(\theta)$ and $\eta_n(\theta)$ for $n = 0 \dots N$ we can find $R_{+,n}(\theta)$ and $R_{-,n}(\theta)$ for $n = 0 \dots N$ and, therefore, Z_{2n+1} and Z_{2n+2} for $n = 0 \dots N$ from (8).

Practically, it is useful to make the following change of variables: $e^{\frac{\theta}{2}} = t$. Then the equations (9-10) read

$$\eta(t) = 8\lambda \int_0^\infty ds \frac{t^2 s e^{-\epsilon(s)}}{s^4 + t^4} , \quad (15)$$

$$\epsilon(t) = \log \left(t^k + \frac{1}{t^k} \right) - \frac{2}{\pi} \int_0^\infty ds \frac{t^2 s \log(1 + \eta^2(s))}{s^4 + t^4} . \quad (16)$$

Let us specialize to $k = 1$ for simplicity. One can show that the functions $\epsilon_n(t)$, $\eta_n(t)$ have rather simple structure:

$$\epsilon_n(t) = \sum_{j=0}^{n-1} G_j^{(n)}(t) (\log t)^j, \quad n \geq 2 , \quad (17)$$

$$\eta_n(t) = \sum_{j=0}^n H_j^{(n)}(t) (\log t)^j, \quad n \geq 0 , \quad (18)$$

where $G_j^{(n)}(t)$ are rational functions with poles allowed at the roots of $t^4 - 1$ and $H_j^{(n)}(t)$ are rational functions with poles allowed at the roots of $t^4 + 1$. This observation allows easy calculation of the integrals (15-16) order by order in λ with the help of the residue theorem and the following formula:

$$\int_0^\infty dt C(t) (\log t)^j = -\frac{(2\pi i)^j}{j+1} \int_\gamma dt C(t) B_{j+1} \left(\frac{\log t}{2\pi i} \right) , \quad (19)$$

where in the right hand side $\log t$ has a brunch cut from 0 to $+\infty$, the contour γ goes from $+\infty$ to 0 below the cut and then to $+\infty$ above the cut, $C(t)$ is a rational function and $B_{j+1}(x)$ is Bernoulli polynomial.

Using the described procedure we find²:

$$\begin{aligned}
Z_1 &= \frac{1}{4} , \\
Z_2 &= \frac{-2 + \pi}{16\pi} , \\
Z_3 &= \frac{\pi - 3}{16\pi} , \\
Z_4 &= \frac{-4 - 8\pi + 3\pi^2}{128\pi^2} , \\
Z_5 &= \frac{10 - \pi^2}{256\pi^2} , \\
Z_6 &= \frac{36 - 2\pi - 3\pi^2}{1536\pi^2} , \\
Z_7 &= \frac{-42 + 126\pi + 49\pi^2 - 27\pi^3}{9216\pi^3} , \\
Z_8 &= \frac{-96 + 96\pi + 64\pi^2 - 27\pi^3}{18432\pi^3} , \\
Z_9 &= \frac{12 - 96\pi - 20\pi^2 + 5\pi^4}{32768\pi^4} , \\
Z_{10} &= \frac{1200 - 2400\pi - 1400\pi^2 + 226\pi^3 + 135\pi^4}{1474560\pi^4} , \\
Z_{11} &= \frac{-660 + 23100\pi - 12100\pi^2 - 25300\pi^3 - 6303\pi^4 + 4725\pi^5}{29491200\pi^5} , \\
Z_{12} &= \frac{-720 + 3600\pi + 1200\pi^2 - 2560\pi^3 - 1536\pi^4 + 675\pi^5}{7372800\pi^5} , \\
Z_{13} &= \frac{4680 - 561600\pi + 978900\pi^2 + 655200\pi^3 + 10114\pi^4 - 30375\pi^6}{4246732800\pi^6} , \\
Z_{14} &= \frac{141120 - 1693440\pi + 764400\pi^2 + 1764000\pi^3 + 625436\pi^4 - 162882\pi^5 - 70875\pi^6}{14863564800\pi^6} , \\
Z_{15} &= \frac{-2520 + 1076040\pi - 4024860\pi^2 - 1425900\pi^3 + 2429714\pi^4 + 2860522\pi^5 + 527265\pi^6 - 509355\pi^7}{55490641920\pi^7} , \\
Z_{16} &= \frac{-40320 + 1128960\pi - 1599360\pi^2 - 1646400\pi^3 - 238336\pi^4 + 1136128\pi^5 + 663552\pi^6 - 297675\pi^7}{52022476800\pi^7} , \\
Z_{17} &= (85680 - 124750080\pi + 931227360\pi^2 - 303878400\pi^3 - 1054571336\pi^4 - 405544384\pi^5 + \\
&\quad + 45621608\pi^6 + 19348875\pi^8) / (53271016243200\pi^8) , \\
Z_{18} &= (80640 - 5160960\pi + 15617280\pi^2 + 6397440\pi^3 - 10554208\pi^4 - 11079488\pi^5 - \\
&\quad - 2895216\pi^6 + 1060922\pi^7 + 385875\pi^8) / (1479750451200\pi^8) , \\
Z_{19} &= (-287280 + 1414279440\pi - 20169928800\pi^2 + 24409032480\pi^3 + 31396649256\pi^4 + 1177819272\pi^5 - \\
&\quad - 19209555560\pi^6 - 17783325576\pi^7 - 2533741371\pi^8 + 3094331625\pi^9) / (5753269754265600\pi^9) .
\end{aligned} \tag{20}$$

From (2), (6) we obtain our main results (1).

3 Numerical Applications

It is easy to check numerically (cf. [14, 16]) that the exact results obtained in the previous section are in agreement with the Airy function asymptotics. According to [15, 10, 14] the perturbative part of the partition function for $k = 1$ is given by

²This result agrees perfectly with the numerical result for $N \leq 16$ in [16].

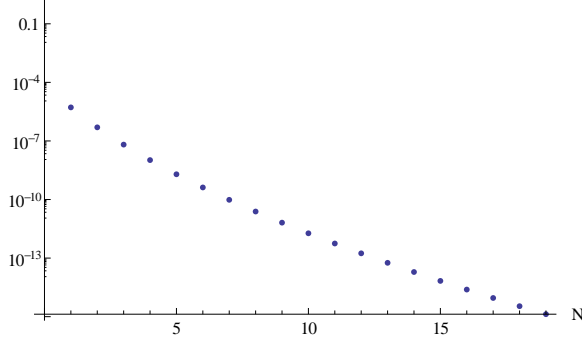


Figure 1: In this figure, the dots represent the sequence $Z(N)/Z^{(\text{pert})}(N) - 1$.

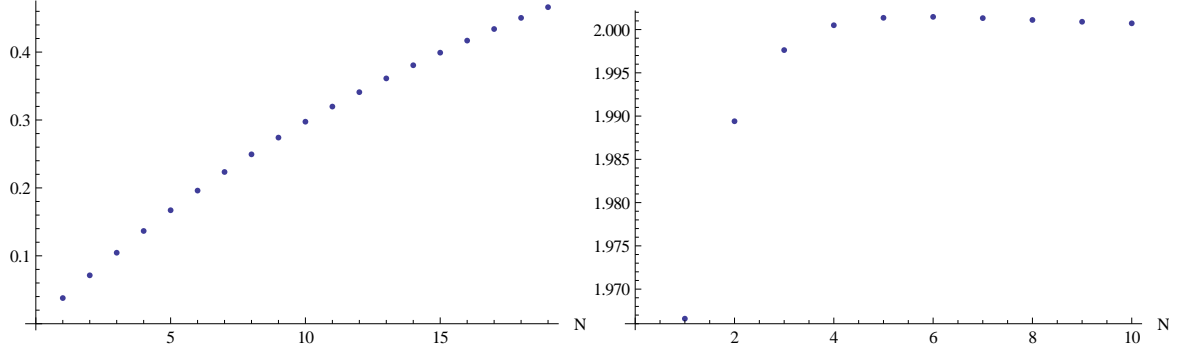


Figure 2: In these figures, the dots represent the sequence (24) (left) and its 9-th Richardson-like transform $\tilde{F}^{(9)}(N)$ (right).

$$Z^{(\text{pert})}(N) = C e^A \text{Ai} \left[C \left(N - \frac{1}{3} - \frac{1}{24} \right) \right], \quad (21)$$

where

$$C = \frac{\pi^{2/3}}{2^{1/3}}, \quad (22)$$

$$A = -\frac{\zeta(3)}{8\pi^2} + \frac{1}{6} \log \frac{\pi}{2} + 2\zeta'(-1) + \frac{1}{2} \log 2 - \frac{1}{3} \int_0^\infty dx \frac{1}{e^x - 1} \left(\frac{3}{x^3} - \frac{1}{x} - \frac{3}{x (\sinh x)^2} \right). \quad (23)$$

The Fig. 1 shows that indeed $Z(N)$ approaches $Z^{(\text{pert})}(N)$ exponentially fast. In [16] it was checked that the non-perturbative part $Z^{(\text{np})} \equiv Z - Z^{(\text{pert})}$ is suppressed by $e^{-2\pi\sqrt{2N}}$ which agrees with the previous analytical results [6, 10, 9]. One can go further and find the leading behavior of the prefactor. Namely, let us consider the following sequence:

$$\tilde{F}(N) \equiv \frac{Z^{(\text{np})}(N)}{Z^{(\text{pert})}(N)} \Big/ N e^{-2\pi\sqrt{2N}} = \left(\frac{Z(N)}{Z^{(\text{pert})}(N)} - 1 \right) \Big/ N e^{-2\pi\sqrt{2N}}. \quad (24)$$

From the previous works one expects that this sequence has an asymptotic expansion of the following form:

$$\tilde{F}(N) = c_0 + \frac{c_1}{N^{1/2}} + \frac{c_2}{N} + \frac{c_3}{N^{3/2}} + \dots \quad (25)$$

This assumption will be verified numerically *a posteriori*. The graph of $\tilde{F}(N)$ is shown on the left of Fig. 2. One can accelerate convergence of the sequence $\tilde{F}(N)$ by performing Richardson-like transforms. Let us define the Richardson-like transform R_γ of a sequence $S(N)$ as

$$R_\gamma[S(N)] \equiv (N/\gamma + 1) S(N + 1) - NS(N)/\gamma . \quad (26)$$

Its crucial property is that

$$R_\gamma \left[c + O\left(\frac{1}{N^\gamma}\right) \right] = c + o\left(\frac{1}{N^\gamma}\right) . \quad (27)$$

In particular, if we define

$$\tilde{F}^{(n)}(N) \equiv R_{\frac{n}{2}} \left[\tilde{F}^{(n-1)}(N) \right] \quad (n \geq 1) , \quad \tilde{F}^{(0)}(N) \equiv \tilde{F}(N) , \quad (28)$$

then one can show that

$$\tilde{F}^{(n)}(N) = c_0 + O\left(\frac{1}{N^{\frac{n+1}{2}}}\right) . \quad (29)$$

The graph of $\tilde{F}^{(9)}(N)$ is shown on the right of Fig. 2. The sequence converges very fast, which verifies the self-consistency of the assumption (25). Our numerical result suggests that $c_0 = 2$ exactly. One can also numerically obtain c_1, c_2, \dots using similar techniques. On the M-theory side of the AdS/CFT correspondence this gives the 1-instanton contribution from M2-branes³:

$$\frac{Z^{(1\text{-inst})}}{Z^{(\text{pert})}} = \left(2N + O(\sqrt{N}) \right) e^{-2\pi\sqrt{2N}} . \quad (30)$$

It would be interesting to check this by a direct calculation of 1-instanton contribution in M-theory. Let us note that the prefactor in (30) cannot be obtained by previously developed techniques since they provide the non-perturbative part of the partition function as non-trivial asymptotic expansions either for large k [6] or for small k [10], whereas the result (30) is for $k = 1$.

Acknowledgments

The authors would like to thank Aspen Center for Physics (NSF Grant No. 1066293) for hospitality. P. P. is supported by the Fonds National Suisse, subsidies 200020-126817 and 200020-137523, and by FASI RF 14.740.11.0347.

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³The smallest instanton action is twice the action of the D2-brane considered in [9] for $k = 1$ because there is no M2 in S^7 which is a degree one cover of the D2-brane wrapping $\mathbb{RP}^3 \in \mathbb{CP}^3$. However, there is an M2 wrapping $S^3 \in S^7$ which is a degree two cover.

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